

Multidimensional Phase Space and Sunset Diagrams

A. Bashir¹, R. Delbourgo² and M.L. Roberts³

School of Mathematics and Physics, University of Tasmania, Hobart, AUSTRALIA 7001

Abstract

We derive expressions for the phase-space of a particle of momentum p decaying into N particles, that are valid for any number of dimensions. These are the imaginary parts of so-called ‘sunset’ diagrams, which we also obtain. The results are given as a series of hypergeometric functions, which terminate for odd dimensions and are also well-suited for deriving the threshold behaviour.

I. Introduction

With so much attention focussed on the properties of branes embedded in higher dimensions, it is of interest to examine the way in which the phase space changes as the space-time is enlarged, since this is what primarily determines the statistical properties of multiparticle systems; only when amplitudes are accentuated or suppressed along certain directions in space is there a pronounced effect on the phase space. This paper is devoted to the topic of multidimensional phase space. We assume that extra dimensions are spatial—though one may envisage that time-like coordinates can be handled by Euclidean continuation—and that all particles propagate into the bulk. Not only that, we suppose that the space-time is flat so that we can neglect topological aspects. If conditions arise that constrain the particles to some subspace, then one may obtain the reduced phase space by altering the number $D - 1$ of spatial dimensions rather trivially; or else if topological effects become important we may be able to sum over discrete modes along flat directions. Thus in almost every respect we are simply investigating the dimensional continuation of standard 4-D expressions for the decay of a system carrying momentum p into *any number* N of particles.

There has been much early work in this area [1], but it is mostly in the context of four dimensions (or lower) and has often been confined to few particles (like two, three or four). The most pertinent recent results in this connection are those of Groote *et al* [2] and of Davydychev and Smirnov [3] because they are the most general. Our own results apply to any value of N and D and are also singularly well-suited for deriving threshold expansions as $\sqrt{p^2} \rightarrow (m_1 + m_2 + \cdots + m_N)$. Because phase space ρ is nothing but the imaginary part of the ‘sunset diagram’ for $p \rightarrow p_1 + p_2 + \cdots + p_N$, our procedure will be to start with this sort of diagram and then obtain ρ_N from its discontinuity in p^2 as we cross the threshold. (Pseudothreshold singularities of these diagrams exist too but are not relevant for physical phase space.)

In Section II, we give the most convenient Feynman parametric description of the general diagram for any dimension $D = 2\ell$ and show how one may readily derive the leading threshold behaviour by expanding about the minimum of the combined denominator; although one may in principle derive the next to leading behaviour by this method, and so on, we do not pursue this parametric approach any further, as there is a much better way of handling the problem, which is described in the following sections. First, in section III, we derive some special cases of the results (small N) by making use of phase space recurrence methods [1] and next show how they may more easily be found by Fourier transformation of the propagator products in section IV. This is our cue

¹ Permanent address: University of Michoacan, Morelia, MEXICO (E-mail: adnan@itzel.ifm.umich.mx)

² E-mail: Bob.Delbourgo@utas.edu.au

³ E-mail: Martin.Roberts@utas.edu.au

for tackling the most general situation, but because the form of our expressions makes it tricky to take the massless limit of any individual particle we specifically suppose that M of the particles are massless and $N - M$ are massive. Our results, stated in section V, are given as a series of hypergeometric terms having the form (hereafter F stands for the usual ${}_2F_1$ function)

$$\rho_N \propto \sum_{j=M(3/2-\ell)+2N(\ell-1)-\ell+1/2} d_j(p^2 - \sigma^2)^j F(a, b; c; 1 - p^2/\sigma^2),$$

where σ is the sum of the masses. An agreeable property of this expansion is that it terminates for odd D and is tailored for deriving the threshold behaviour.

II. Feynman parametric form

In 2ℓ dimensions, sunset diagrams with N internal lines, produce integrals of the type⁴,

$$I_N(p; \{\nu\}) = i \left(\prod_{i=1}^N \int \frac{i\Gamma(\nu_i)}{(p_i^2 - m_i^2)^{\nu_i}} \bar{d}^{2\ell} p_i \right) \bar{\delta}^{2\ell}(p - \sum_i p_i). \quad (1)$$

We have included a Gamma function in each numerator since the propagator powers usually arise via momentum or mass derivatives of the case $\nu = 1$, and it simplifies the subsequent expressions. By attaching a Feynman parameter α to each internal line, one may combine denominators in the standard fashion and integrate over the internal momenta to establish⁵ that

$$I_N(p; \{\nu\}) = \frac{(-1)^N}{(-4\pi)^{(N-1)\ell}} \left(\prod_{i=1}^N \int_0^1 \frac{d\alpha_i}{\alpha_i^{\nu_i-\ell+1}} \right) \frac{\Gamma(\sum_i \nu_i + \ell - N\ell) \delta(1 - \sum_i \alpha_i)}{[p^2 - \sum_i (m_i^2/\alpha_i)]^{\sum_i \nu_i + \ell - N\ell}}, \quad (2)$$

but it is easier to prove the result by induction in fact. Suppose that (2) is true for N ; the case $N + 1$ represents a further convolution:

$$I_{N+1}(p; \{\nu\}) = i \int \bar{d}^{2\ell} k I_N(k; \{\nu\}) \Gamma(\nu_{N+1}) / [(k + p)^2 - m_{N+1}^2]^{\nu_{N+1}}$$

so introduce an extra Feynman parameter β . Combining the new denominator with (2) and integrating over intermediate loop momentum k , one remains with

$$I_{N+1}(p) = \frac{(-1)^{N+1}}{(-4\pi)^{N\ell}} \left(\prod_{i=1}^N \int_0^1 \frac{d\alpha_i}{\alpha_i^{\nu_i-\ell+1}} \right) \int_0^1 d\beta \frac{\Gamma(\sum_{i=1}^{N+1} \nu_i - N\ell) \delta(1 - \sum_i \alpha_i) (1 - \beta)^{\nu_{N+1}-1} \beta^{\ell-\nu_{N+1}-1}}{[p^2(1 - \beta) - \sum_i (m_i^2/\alpha_i) - m_{N+1}^2(1 - \beta)/\beta]^{\sum_i \nu_i - N\ell + \nu_{N+1}}}.$$

Now all one need do is rescale $\alpha_i = \beta_i/(1 - \beta)$ for $i = 1$ to N and call $\beta \equiv \beta_{N+1}$. A final relabelling of $\beta \rightarrow \alpha$ reproduces (2) for $N \rightarrow N + 1$. Since we know the result (2) is correct for $N = 2$ —it is very familiar to graphologists as the simplest self-energy calculation—we have thereby proved the result inductively for any N .

The singularities of $I_N(p)$ in p^2 will arise when p^2 equals the combination $M^2(\alpha) \equiv \sum_{i=1}^N (m_i^2/\alpha_i)$, so let us examine the behaviour of $M^2(\alpha)$ in the region of integration $0 \leq \alpha_i \leq 1$, subject to the condition $\sum_i \alpha_i = 1$. For definiteness, we shall assume for the rest of this section that all masses are nonzero. By introducing a Lagrangian multiplier for the last constraint, it is easy to see that $M^2(\alpha)$ is minimised in the physical region when the Feynman parameters equal $\alpha_{i0} \equiv m_i / \sum_{j=1}^N m_j \equiv m_i / \sigma$,

⁴The bar notation means: division by 2π when integrating over each momentum component and a factor of 2π for each delta function.

⁵This involves some finicky rescaling of the Feynman parameters. See reference [4] for the case $N = 3$.

whereupon $M^2(\alpha_0) = (\sum_{j=1}^N m_j)^2 = \sigma^2$. Other values of α_i are possible (by reversing one or other of the signs of α_{i0}) but they correspond to pseudothresholds and lie outside the integration region. It is then clear that the magnitude of $I_N(p)$ will be dominated by α_i values in the vicinity of α_{i0} , so it is sensible to expand about these minima if we want to determine the leading behaviour near threshold.

This procedure can be developed systematically (see the Appendix for the example $N = 2$ with $\nu_1 = \nu_2 = 1$) but as there is an alternative and preferable way of tackling the problem, which we defer to the following sections, here we shall simply extract the leading threshold behaviour, because this can be done rather quickly and painlessly. Begin with (2) and note that

$$M^2(\alpha) = \sigma^2 + \sum_i (\alpha_i - \alpha_{i0})^2 \sigma^3 / m_i - \sum_i (\alpha_i - \alpha_{i0})^3 \sigma^4 / m_i^2 + \dots$$

One sees that for $1 - p^2/\sigma^2 \equiv \Delta^2$ small the integral is dominated by values of α near the minimum α_0 . Therefore write $\alpha_i = \alpha_{i0} + \Delta \xi_i$, so that the denominator of (2) reads

$$-\Delta^2 \sigma^2 [1 + \sigma \sum_i \xi_i^2 / m_i + \Delta \sigma^2 \sum_i \xi_i^3 / m_i^2 + O(\Delta^2)].$$

Since the integrals in α can be made to run between 0 and ∞ , because of the delta-function constraint, the integral over ξ_i runs from $-m_i/\sigma\Delta$ to ∞ . So, in the limit as $\Delta \rightarrow 0$, we can take ξ_i to run from $-\infty$ to ∞ (the correction to the integral is exponentially small as $\exp(-1/\Delta^2)$) and expand the product of the α_i about the minimum. Assuming all masses are non-zero (see later sections for a relaxation of this condition) we are thereby led to the leading behaviour,

$$I_N(p, \{\nu\}) = c_{N\ell} \Delta^{(N-1)(2\ell+1)-2} \sum_i \nu_i \left(\prod_i m_i^{\ell-\nu_i-1} \right) / (4\pi)^{\ell(N-1)} \sigma^{(2-N)\ell-N+\sum_i \nu_i}, \quad (3)$$

where

$$c_{N\ell} \simeq \left(\prod_i \int d\xi_i \right) (-1)^{N-\sum_i \nu_i} \frac{\delta(\sum_i \xi_i) \Gamma(\sum_i \nu_i - (N-1)\ell)}{[1 + \sigma \sum_i \xi_i^2 / m_i]^{\sum_i \nu_i - (N-1)\ell}}.$$

If one now represents the delta function and the denominator by integrals, the coefficient $c_{N\ell}$ can be explicitly evaluated as follows:

$$\begin{aligned} c_{N\ell} &= \frac{1}{2\pi} \left(\prod_i \int d\xi_i \right) \int_{-\infty}^{\infty} dk \int_0^{\infty} d\alpha \alpha^{\sum_i \nu_i - 1 - (N-1)\ell} e^{-\alpha - k^2/4\alpha} \\ &\quad \times (-1)^{N-\sum_i \nu_i} \prod_i \exp \left[-\frac{\alpha\sigma}{m_i} \left(\xi_i - \frac{ikm_i}{2\alpha\sigma} \right)^2 \right] \\ &= (-1)^{N-\sum_i \nu_i} \varpi^{1/2} \pi^{(N-1)/2} \sigma^{-N/2} \Gamma(\sum_i \nu_i - (\ell + 1/2)((N-1))); \quad \varpi \equiv \prod_i m_i. \end{aligned}$$

So, all told, the leading threshold behaviour is dominated by

$$I_N(p, \{\nu\}) = (-1)^{N-\sum_i \nu_i} \frac{\pi^{(N-1)/2}}{(4\pi)^{(N-1)\ell}} \frac{\Gamma(\sum_i \nu_i - (N-1)(\ell + 1/2)) \prod_i m_i^{\ell-\nu_i-1/2}}{\sigma^{(2-N)\ell-N/2+\sum_i \nu_i}} \Delta^{(N-1)(2\ell+1)-2} \sum_i \nu_i. \quad (4)$$

The case of greatest interest is $\nu_i = 1$, all i , when (4) reduces to

$$I_N(p) = \frac{\pi^{(N-1)/2}}{(4\pi)^{(N-1)\ell}} \frac{\Gamma((N-1)/2 - \ell(N-1))}{\varpi^{3/2-\ell} \sigma^{N/2+\ell(2-N)}} (1 - p^2/\sigma^2)^{\ell(N-1)-(N+1)/2}. \quad (5)$$

We may then continue this expression above threshold ($p^2 \geq \sigma^2$), in order to obtain the behaviour of the N -body phase space as the discontinuity:

$$\rho_N(p) = 2\mathfrak{S}I_N(p) \simeq \frac{\pi^{(N+1)/2}}{(4\pi)^{(N-1)\ell}} \frac{\varpi^{\ell-3/2}}{\sigma^{N/2+\ell(N-1)}} \frac{(p^2/\sigma^2 - 1)^{\ell(N-1)-(N+1)/2}}{\Gamma((N-1)(\ell-1/2))} \quad (6)$$

This result agrees with the answer obtained by Davydychev and Smirnov [3] and reduces to the well-known four-dimensional behaviour found by Hagedorn and Almgren [1], when we set $\ell = 2$, namely

$$\rho_N(p) \sim Q^{(3N-5)/2},$$

where $Q = \sqrt{p^2} - \sigma$ is the energy release. However (6) supplies the answer in any dimension with the appropriate coefficient.

III. Exact results

We now consider phase space in general and derive some precise results for any ℓ, N as they arise from recurrence relations between phase space expressions for smaller N and not necessarily around threshold. Begin with the definition of the N -body phase space integral,

$$\rho_N(p) \equiv \rho_{p \rightarrow 1+2+\dots+N} \equiv \prod_i \left(\int d^2 p_i \theta(p_i) \delta(p_i^2 - m_i^2) \right) \delta^2(p - \sum_{i=1}^N p_i). \quad (7)$$

The measures $d^{2\ell-1}\vec{p} = |\vec{p}|^{2\ell-2} d|\vec{p}| \cdot (\sin \theta)^{2\ell-3} d\theta \cdot 2\pi^{\ell-1} / \Gamma(\ell-1) = |\vec{p}|^{2\ell-2} d|\vec{p}| \cdot 2\pi^{\ell-1/2} / \Gamma(\ell-1/2)$, come in useful if one were able to integrate over angles. Thus the two-body result is readily evaluated in this way to be

$$\rho_2(p) = \frac{\pi(4\pi)^{1/2-\ell} q^{2\ell-3}}{\Gamma(\ell-1/2) \sqrt{p^2}},$$

where q is the centre of mass spatial momentum, so that $\sqrt{q^2 + m_1^2} + \sqrt{q^2 + m_2^2} = \sqrt{p^2}$. Tidying up, the result can be expressed covariantly as

$$\rho_{p \rightarrow 1+2} = \frac{\pi^{1-\ell} \Gamma(\ell-1) \lambda^{2\ell-3}}{2^{2\ell-1} (p^2)^{\ell-1} \Gamma(2\ell-2)}; \quad \lambda \equiv \sqrt{p^4 + m_1^4 + m_2^4 - 2m_1^2 m_2^2 - 2p^2 m_1^2 - 2p^2 m_2^2}. \quad (8)$$

The three-body phase space can also be evaluated by brute force methods and reduced to a triangular integral over three Mandelstam variables:

$$\rho_{p \rightarrow 1+2+3} = \frac{2\pi(p^2)^{1-\ell}}{(4\pi)^{2\ell} \Gamma(2\ell-2)} \int \int \int ds dt du \delta(s+t+u - m_1^2 - m_2^2 - m_3^2 - p^2) [\Phi(s, t, u)]^{\ell-2} \theta(\Phi),$$

where $\Phi(s, t, u)$ is the Kibble [5] cubic (simply a Gram determinant),

$$\begin{aligned} \Phi(s, t, u) &\equiv stu - s(m_2^2 m_3^2 + p^2 m_1^2) - t(m_3^2 m_1^2 + p^2 m_2^2) - u(m_1^2 m_2^2 + p^2 m_3^2) \\ &\quad + 2(m_1^2 m_2^2 m_3^2 + p^2 m_1^2 m_2^2 + p^2 m_2^2 m_3^2 + p^2 m_3^2 m_1^2) = -p^2 \lambda^2 (|\vec{p}_1|^2, |\vec{p}_2|^2, |\vec{p}_3|^2). \end{aligned}$$

By changing variables to s and $t - u$, one integration may be performed and the problem reduced to the single integral [6],

$$\rho_{p \rightarrow 1+2+3} = \frac{(32\pi)^{2-2\ell}}{(\Gamma(\ell-1/2))^2 (p^2)^{\ell-1}} \int_{(m_1+m_2)^2}^{(\sqrt{p^2}-m_3)^2} s^{1-\ell} \mathcal{D}^{\ell-3/2} ds, \quad (9)$$

where $\mathcal{D} = [s - (m_3 + \sqrt{p^2})^2][s - (m_3 - \sqrt{p^2})^2][s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]$. When we are in four dimensions ($\ell = 2$) the integral is nothing but the area within the Dalitz plot but its evaluation for arbitrary masses is easier said than done, because (9) is an elliptic function in general! However in three dimensions the integration over the region becomes possible because it can be converted into

$$\rho_3(p) \rightarrow \frac{1}{32\pi\sqrt{p^2}} \int_{(m_1+m_2)^2}^{(\sqrt{p^2}-m_3)^2} \frac{ds}{\sqrt{s}} = \frac{1}{16\pi} \left(1 - \frac{m_1 + m_2 + m_3}{\sqrt{p^2}} \right).$$

In fact the three-body phase space integration is tractable for all odd D dimensions, because one remains with a polynomial in half-integral powers of s which is readily handled [7]. Two other cases are amenable to an exact treatment in terms of ‘elementary’ functions for any ℓ and $N = 3$, namely (i) two masses are set to zero, or (ii) one mass vanishes and the two other masses are equal. In case (i) put $m_2 = m_3 = 0, m_1 = m$ so $\mathcal{D} \rightarrow (s - p^2)^2(s - m^2)^2$ whereupon the 3-body phase space reads

$$\rho_3(p) \rightarrow \frac{\pi^{1-2\ell}(p^2/m^2 - 1)^{4\ell-5}[\Gamma(\ell-1)]^2}{2^{4\ell-1}(m^2)^{3-2\ell}\Gamma(4\ell-4)} F(3\ell-3, 2\ell-2; 4\ell-4; 1 - \frac{p^2}{m^2}) \theta(p^2 - m^2),$$

while in case (ii) put $m_3 = 0, m_1 = m_2 = m$ so that $\mathcal{D} \rightarrow s(s - p^2)^2(s - 4m^2)$ and

$$\rho_3(p) \rightarrow \frac{\pi^{3/2-2\ell}\Gamma(\ell-1)}{2^{6\ell-3}\Gamma(3\ell-5/2)} \frac{(p^2 - 4m^2)^{3\ell-7/2}}{(p^2)^{\ell-1}m} F(\frac{1}{2}, \ell - \frac{1}{2}, 3\ell - \frac{5}{2}; 1 - \frac{p^2}{4m^2}) \theta(p^2 - 4m^2).$$

Of course we can also proceed to the limit $m \rightarrow 0$ in either case. Any other set of masses produces ‘elliptic’ results [1, 6] for even D or integral ℓ .

A more systematic way of arriving at Lorentz invariant answers, without resorting to integrations over spatial momenta in the standard approach and interpreting them covariantly, is to apply Almgren’s method. In that method one partitions the set of outgoing particles into subsets (call them A, B etc.) and obtains ρ via mass integration convolutions; for instance the three-body result (9) can be construed as

$$\rho_{p \rightarrow 1+2+3} = \int \rho_{p \rightarrow A+3} \rho_{A \rightarrow 1+2} \bar{d}m_A^2,$$

where s is interpreted as the intermediate m_A^2 . More generally, one may evaluate the N -body phase space through the double integral

$$\rho_{p \rightarrow 1+2+\dots+j+(j+1)\dots+N} = \int \rho_{p \rightarrow A+B} \rho_{A \rightarrow 1+2+\dots+j} \rho_{B \rightarrow (j+1)+\dots+N} \bar{d}m_A^2 \bar{d}m_B^2. \quad (10)$$

Although this is quite a satisfactory numerical way of calculating ρ , it does not shed a great deal of analytical light on the nature of the problem; but it does serve as a nice check of analytical results obtained in a different manner, which we will now exhibit.

IV. Coordinate space method

The sunset diagram having all $\nu=1$ is nothing but the 2ℓ -dimensional Fourier transform of the product of N causal functions:

$$I_N(p) = -i \int d^{2\ell}x \exp(ip.x) \prod_{i=1}^N [i\Delta_c(x|m_i)], \quad (11)$$

and the phase space integral is simply given by

$$\rho_{p \rightarrow 1+2+\dots+N} = 2\Im I_N(p),$$

which is nonvanishing for $p^2 \geq (m_1 + m_2 + \dots + m_N)^2$. Of course (11) is easier stated than done except in the simplest cases (like $N = 2$ or 3) because the causal function is a Bessel function in general [8],

$$i\Delta_c(x|m) = \frac{1}{(2\pi)^\ell} \left(\frac{m}{r}\right)^{\ell-1} K_{\ell-1}(mr); \quad r \equiv \sqrt{-x^2 + i\epsilon}.$$

In the massless limit, when $m \rightarrow 0$, the sunset diagram reduces to a “superpropagator” with integer index since

$$i\Delta_c(x|0) \equiv iD_c(x) = \Gamma(\ell - 1)/4\pi^\ell r^{2\ell-2}.$$

However, as noted and indeed emphasized by Berends *et al* [9] and by Groote *et al* [2], one can make considerable progress using coordinate space methods in the odd-dimensional massive cases, because the Bessel function reduces to an exponential times a polynomial.

Let us begin systematically by considering the most trivial case *when all intermediate masses vanish*. Carrying out the angular integration in (11), the sunset integral reduces to

$$I_N(p) \rightarrow -(2\pi)^\ell q^{1-\ell} \int_0^\infty dr \, r^\ell J_{\ell-1}(qr) \left[\frac{\Gamma(\ell-1)}{4\pi^\ell r^{2\ell-2}} \right]^N; \quad q^2 \equiv -p^2, \quad (12)$$

which can be evaluated straightforwardly [10]. Thus one finds that

$$\lim_{m_i \rightarrow 0} I_N(p) = -(4\pi)^{\ell(1-N)} \frac{[\Gamma(\ell-1)]^N \Gamma(\ell + N - N\ell)}{\Gamma(N\ell - N)} (-p^2 - i\epsilon)^{N\ell - \ell - N}, \quad (13)$$

so the massless phase space integral reduces, in 2ℓ dimensions, to⁶

$$\lim_{m_i \rightarrow 0} \rho_N(p) = 2\Im I_N(p) = \frac{(4\pi)^{1+\ell-N\ell} [\Gamma(\ell-1)]^N}{2\Gamma(N\ell - N)\Gamma((N-1)(\ell-1))} (p^2)^{N\ell - \ell - N} \theta(p^2). \quad (14)$$

Both (13) and (14) are exact too. One may readily check that the cases $N = 1, 2, 3$ produce the correct answers, by taking appropriate limits of earlier results. Also it is very instructive to check that the Almgren recurrence formulae, such as (10), are properly obeyed, not only in their momentum dependence, but in their multiplicative coefficients.

Now it turns out that the integral of two Bessel functions with an exponential can also be handled [10]. Therefore one can improve on above and calculate analytically the sunset integral for one particle massive (m) and the remaining $N-1$ particles massless. In those circumstances one is led to [11]

$$\begin{aligned} I_N(p) &= -\left(\frac{q}{m}\right)^{1-\ell} \int_0^\infty dr \, r^{2(N-1)(1-\ell)+1} J_{\ell-1}(qr) K_{\ell-1}(mr) \cdot \left(\frac{\Gamma(\ell-1)}{4\pi^\ell}\right)^{N-1} \\ &= -\frac{[\Gamma(\ell-1)]^{N-1} \Gamma(\ell + N - N\ell) \Gamma(N - N\ell + 2\ell - 1)}{2^{2\ell(N-1)} \pi^{\ell(N-1)} (m^2)^{N+\ell-N\ell} \Gamma(\ell)} F(\ell + N - N\ell, N - N\ell + 2\ell - 1; \ell; \frac{p^2}{m^2}). \end{aligned} \quad (15)$$

Noting that in respect of the variable z the hypergeometric function $F(a, b; c; z)$ for real a, b, c has a branch point at $z = 1$ and that the discontinuity across the cut (which runs to $+\infty$) is

$$\Im F(a, b; c; z) = -\frac{\pi \Gamma(c)(z-1)^{c-a-b} \theta(z-1)}{\Gamma(a)\Gamma(b)\Gamma(1-a-b+c)} F(c-a, c-b; c-a-b+1; 1-z), \quad (16)$$

⁶Here and afterwards, we use the fact that $\Im \Gamma(a)(-k^2 + i\epsilon)^a = -\pi |k^2|^a \theta(k^2)/\Gamma(1-a)$ for real a .

we deduce the phase space result,

$$\rho_N(p) = \frac{\pi^{1-\ell(N-1)} [\Gamma(\ell-1)]^{N-1} (p^2 - m^2)^{2N\ell-2\ell-2N+1} \theta(p^2 - m^2)}{2^{2\ell(N-1)-1} (m^2)^{N\ell-N-\ell+1} \Gamma(2(N-1)(\ell-1))} F(N(\ell-1), (N-1)(\ell-1); 2(N-1)(\ell-1); 1 - \frac{p^2}{m^2}) \quad (17)$$

The pair of expressions (16) and (17) are precise as well. Contrast (17) with the expression found by Beneke and Smirnov [12] for a one-loop vertex diagram with two massive and one massless particle. By taking the limit $m \rightarrow 0$ of (16) and (17) and suitably manoeuvring the hypergeometric function⁷ one can show that they collapse into (13) and (14). Additionally we may deduce the particular cases $N = 2, 3$ previously by direct substitution; for instance, in four dimensions ($\ell = 2$), one gets

$$\rho_N(p) = \frac{2\pi(p^2 - m^2)^{2N-3}}{(16\pi^2)^{N-1} (m^2)^{N-1} \Gamma(2N-2)} F(N, N-1; 2(N-1); 1 - \frac{p^2}{m^2}) \theta(p^2 - m^2),$$

so the three-body phase space with a single massive particle ($p^2 \geq m^2$) brings in a logarithmic function:

$$\rho_3(p) = \frac{(p^2 - m^2)^3}{768\pi^3 m^4} F(3, 2; 4; 1 - \frac{p^2}{m^2}) = \frac{m^2}{256\pi^3 p^2} \left[\left(\frac{p^4}{m^4} - 1 \right) - 2 \frac{p^2}{m^2} \ln\left(\frac{p^2}{m^2}\right) \right].$$

V. Hypergeometric Expansions

As soon as we have at least two massive particles, we come across integrals involving products of three Bessel functions with different arguments and a power of r . This is not given in the standard texts⁸, though it is surely some generalization of a hypergeometric function for we definitely know how to evaluate the $N = 2$ case directly in momentum space; but for larger N it would seem that the coordinate space method stalls. Only when $p = 0$, which corresponds to a vacuum ‘watermelon diagram’, can the integrations sometimes be carried out [13]. However, we will now present a method which is still suitable for handling the general problem and which works wonderfully well in odd-dimensional spaces. It relies on the observation that the modified Bessel function possesses an asymptotic representation ($\mu \equiv 4\nu^2$),

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \dots \right]$$

that *terminates for ν or ℓ half-integral*, corresponding to D odd. Even for integral ℓ or even dimensions, we shall see that it provides a very nice threshold expansion in terms of hypergeometric functions [14]. To show how this works, assume first that all the intermediate particles are massive. Since

$$i\Delta_c(x|m) = \frac{1}{(2\pi)^\ell} \left(\frac{m}{r} \right)^{\ell-1} \sqrt{\frac{\pi}{2mr}} e^{-mr} \sum_{j=0}^{\infty} \frac{\Gamma(\ell+j-1/2)}{j! \Gamma(\ell-j-1/2)} (2mr)^{-j}, \quad (18)$$

the product of causal Green functions produces the series

$$\prod_{j=1}^N i\Delta_c(x|m_j) = \left(\frac{\pi^{1/2-\ell}}{2^{\ell+1/2} r^{\ell-1/2}} \right)^N \prod_{j=1}^N m_j^{\ell-3/2} \cdot e^{-r \sum_j m_j} \left[1 + \frac{(2\ell-1)(2\ell-3)}{8r} \sum_{j=1}^N \frac{1}{m_j} + \dots \right].$$

⁷e.g. using the relation $F(a, b; c; z) = (1-z)^{-b} F(c-a, b; c; z/(z-1))$.

⁸ The formula $2 \int_0^\infty r^{\nu+1} K_\mu(ar) K_\mu(br) J_\nu(cr) dr = \sqrt{\frac{\pi}{2}} \left(\frac{c}{ab} \right)^{\nu+1} \frac{\Gamma(\nu+\mu+1) \Gamma(\nu-\mu+1)}{\Gamma(\nu+3/2) (1+z)^{\nu+1/2}} F(1/2-\mu, 1/2+\mu; 3/2+\nu; (1-z)/2)$, where $z \equiv (a^2+b^2+c^2)/2ab$, is given in [15].

The leading (and in 3-D the only) term of the rhs above yields the leading behaviour of the sunset diagram,

$$\begin{aligned} I_N^0(p) &= -\frac{(2\pi)^{\ell-N\ell+N/2}}{q^{\ell-1}\varpi^{3/2-\ell}} \int_0^\infty dr r^{\ell-N\ell+N/2} e^{-r\sigma} J_{\ell-1}(qr) \\ &= -\frac{2^{1-N\ell-N/2}\pi^{\ell-N\ell+N/2}\Gamma((2-N)\ell+N/2)}{\Gamma(\ell)\sigma^{(2-N)\ell+N/2}\varpi^{3/2-\ell}} F\left(\frac{2\ell-N\ell+N/2}{2}, \frac{2\ell-N\ell+N/2+1}{2}; \ell; \frac{p^2}{\sigma^2}\right), \end{aligned} \quad (19)$$

where $\varpi \equiv \prod_i m_i$, $\sigma \equiv \sum_i m_i$. The imaginary part then produces the leading contribution to the phase space integral,

$$\begin{aligned} \rho_N^0(p) &= \frac{2^{2\ell(1-N)+1}\pi^{\ell(1-N)+(N+1)/2}(p^2-\sigma^2)^{\ell(N-1)-(N+1)/2}}{\varpi^{3/2-\ell}\sigma^{N\ell-1-N/2}\Gamma(\ell(N-1)-(N+1)/2)} \theta(p^2-\sigma^2) \times \\ &\quad F\left(\frac{N\ell-N/2}{2}, \frac{N\ell-1-N/2}{2}; \ell(N-1)-\frac{N-1}{2}; 1-\frac{p^2}{\sigma^2}\right). \end{aligned} \quad (20)$$

It should be noted that expressions (19) and (20) represent the *complete* results for 3-D, when they reduce respectively to

$$I_N(p) \rightarrow -\frac{\pi^{3/2-N}\Gamma(3-N)}{2^{2N-2}\sigma^{3-N}\sqrt{\pi}} F\left(\frac{3-N}{2}, \frac{4-N}{2}; \frac{3}{2}; \frac{p^2}{\sigma^2}\right)$$

and

$$\rho_N(p) \rightarrow \frac{\pi^{2-N}(p^2-\sigma^2)^{N-2}}{2^{3N-4}\sigma^{N-1}\Gamma(N-1)} F\left(\frac{N}{2}, \frac{N-1}{2}; N-1; 1-\frac{p^2}{\sigma^2}\right) = \frac{\pi(\sqrt{p^2}-\sigma)^{N-2}\theta(p^2-\sigma^2)}{(4\pi)^{N-1}\Gamma(N-1)\sqrt{p^2}}.$$

Nonleading behaviours (for values of $\ell \neq 3/2$) of I_N and ρ_N can be found from expansion (18); for instance to the next order we encounter the terms,

$$\begin{aligned} I_N^1(p) &= \frac{(2\ell-1)(2\ell-3)\pi^{\ell((1-N)+N/2)}\Gamma(2\ell-1-N\ell+N/2)}{2^{2+N\ell+N/2}\varpi^{3/2-\ell}\sigma^{\ell(2-N)+N/2-1}\Gamma(\ell)\mu} \times \\ &\quad F\left(\frac{2\ell-N\ell+N/2-1}{2}, \frac{2\ell-N\ell+N/2}{2}; \ell; \frac{p^2}{\sigma^2}\right), \end{aligned} \quad (21)$$

$$\begin{aligned} \rho_N^1(p) &= \frac{(2\ell-1)(2\ell-3)\pi^{\ell(1-N)+(N+1)/2}(p^2-\sigma^2)^{\ell(N-1)+(1-N)/2}}{2^{2\ell(N-1)+3}\varpi^{3/2-\ell}\sigma^{N(\ell-1/2)}\mu\Gamma(\ell(N-1)+(3-N)/2)} \theta(p^2-\sigma^2) \times \\ &\quad F\left(\frac{N\ell+1-N/2}{2}, \frac{N\ell-N/2}{2}; \ell(N-1)+\frac{3-N}{2}; 1-\frac{p^2}{\sigma^2}\right), \end{aligned} \quad (22)$$

where $1/\mu \equiv \sum_j 1/m_j$; and so on for higher I_N^k, ρ_N^k . The method is entirely systematic and one may proceed to as high an order k as needed. Observe that the resulting series is a sort of threshold expansion because the powers of $(p^2-\sigma^2)$ which multiply $F(a, b; c; 1-p^2/\sigma^2)$ keep on increasing as we raise k while the denominators are associated with sums of the type $\sum_j (1/m_j)^k$.

However these expansions are deficient in one respect: it is very tricky to consider the limit as one or several masses vanishes, because the approximation (18) is fairly useless for massless propagators; in that case we should be using directly $iD_c(x) \propto r^{2-2\ell}$, rather than the asymptotic expansion (18). Finally then we will consider the case where M of the N particles are massive

while the remaining $N - M$ are massless; this covers essentially all cases of interest. The sunset integral is given *ab initio* by

$$I_N(p) = -\frac{(2\pi)^\ell}{q^{\ell-1}} \int_0^\infty dr r^\ell J_{\ell-1}(qr) \left(\frac{\Gamma(\ell-1)}{4\pi^\ell r^{2\ell-2}} \right)^{N-M} \left(\frac{(\pi r)^{1/2-\ell}}{2^{\ell+1/2}} \right)^M e^{-r\sigma} \varpi^{\ell-3/2} \times \left[1 + \frac{(2\ell-1)(2\ell-3)}{8r\mu} + \dots \right], \quad (23)$$

where the symbols now refer simply to the massive intermediate particles; therefore $\varpi \equiv \prod_{j=1}^M m_j$, $\sigma \equiv \sum_{j=1}^M m_j$, $1/\mu \equiv \sum_{j=1}^M (1/m_j)$. The integrals in (23) are readily performed and the leading term of the full answer is

$$I_N^0(p) = -\frac{2^{2(M-N)+1-M\ell-M/2} \pi^{\ell(1-N)+M/2} [\Gamma(\ell-1)]^{N-M} \Gamma(\ell(2-M) + (M-N)(2\ell-2) + M/2)}{\varpi^{3/2-\ell} \sigma^{\ell(2-M)+(M-N)(2\ell-2)+M/2} \Gamma(\ell)} \times F\left(\frac{\ell(2-M)+M/2}{2} + (M-N)(\ell-1), \frac{\ell(2-M)+1+M/2}{2} + (M-N)(\ell-1); \ell; \frac{p^2}{\sigma^2}\right). \quad (24)$$

Taking its discontinuity, the leading phase space behaviour is ($c \equiv M(3/2-\ell) + 2N(\ell-1) - \ell + 1/2$ and $p^2 \geq \sigma^2$ below),

$$\rho_N^0(p) = \frac{2^{2\ell(1-N)+1} \pi^{\ell(1-N)+(M+1)/2} [\Gamma(\ell-1)]^{N-M} (p^2 - \sigma^2)^{M(3/2-\ell)+2N(\ell-1)-(\ell+1/2)}}{\varpi^{3/2-\ell} \sigma^{M(7/2-3\ell)+2(M+N)(\ell-1)-1} \Gamma(c)} \times F\left((\ell-1)(N-M) + \frac{M(2\ell-1)}{4}, (\ell-1)(N-M) + \frac{M(2\ell-1)}{4} - \frac{1}{2}; c; 1 - \frac{p^2}{\sigma^2}\right). \quad (25)$$

All the previous cases fall out of (24) and (25) by making the relevant substitutions for M and ℓ . Of course one may also derive nonleading terms I_N^k, ρ_N^k in exactly the same way as before and they correspond to higher order threshold corrections. Again, these corrections entrain higher powers of $(p^2 - \sigma^2)$ and terminate for half-integral ℓ .

VI. Conclusions

Before we can comprehend the statistical effects of phase space on brane physics, it is vital to understand phase space in flat D -dimensional spacetime for any number of particles, with arbitrary masses. This paper has been devoted to that subject and we have arrived at results for $\rho_N(p)$ and $I_N(p)$ that have culminated in formulae (14), (23), (24) and (25). These comprise the high-energy and low-energy characteristics and at any energy in-between. We believe that these analytical expressions are as compact as one can make them and will turn out to be practically useful. Otherwise one will be obliged to resort to numerical methods.

Acknowledgements

We are grateful to Dr AI Davydychev for supplying us with a comprehensive list of references in this vast topic, for pointing us in the right direction and for valuable comments on our work. We also thank the Australian Research Council for financial support under grant number A00000780.

Appendix. At the threshold of a sunset

In this appendix we show explicitly how the threshold expansion can be carried out in momentum space via Feynman parameters. We illustrate the case $N = 2$ for any dimension 2ℓ to keep the algebra simple, when the sunset integral is just

$$I_2(p) = \frac{\Gamma(2-\ell)}{(-4\pi)^\ell} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \frac{\delta(\alpha_1 + \alpha_2 - 1)}{[p^2 \alpha_1 \alpha_2 - m_1^2 \alpha_2 - m_2^2 \alpha_1]^{2-\ell}}.$$

Never mind the fact that I_2 can be expressed as a linear combination of two hypergeometric functions ($a = \ell - 1, b = 2 - \ell, c = \ell$) with arguments $\frac{1}{2}[(m_1^2 - m_2^2 \pm p^2)/\lambda + 1]$ or that it can be accorded an exact geometrical interpretation [16]; our purpose here is to show how one may arrive at a systematic threshold expansion, by expanding about the minimum of the denominator, regarded as a function of α . Thus change variables to

$$\alpha_i = \frac{m_i}{m_1 + m_2} + \Delta \rho_i, \quad p^2 \equiv (m_1 + m_2)^2 [1 - \Delta^2].$$

The delta-function constraint on α corresponds to $\rho_1 = -\rho_2 \equiv \rho$, whereupon the sunset integral reduces to

$$I_2(p) = \frac{\Gamma(2-\ell)}{(4\pi)^\ell} \Delta^{2\ell-3} \int_{-\frac{m_1}{(m_1+m_2)\Delta}}^{\frac{m_2}{(m_1+m_2)\Delta}} \frac{d\rho}{[m_1 m_2 + \rho^2 (m_1 + m_2)^2 (1 - \Delta^2) - \Delta \rho (m_1^2 - m_2^2)]^{2-\ell}}.$$

The leading behaviour is found by setting $\Delta = 0$ in the denominator of the integrand and extending the limits to $\pm\infty$. Since

$$\int_0^\infty \frac{d\rho}{[m_1 m_2 + \rho^2 (m_1 + m_2)^2]^{2-\ell}} = \frac{(m_1 m_2)^{\ell-3/2}}{2(m_1 + m_2)} \int_0^\infty \frac{dv v^{-1/2}}{(1+v)^{2-\ell}} = \frac{(m_1 m_2)^{\ell-3/2}}{2(m_1 + m_2)} \frac{\sqrt{\pi} \Gamma(3/2 - \ell)}{\Gamma(2 - \ell)},$$

it follows that

$$I_2^0(p) = \frac{(m_1 m_2 \Delta^2)^{\ell-3/2} \sqrt{\pi}}{(4\pi)^\ell (m_1 + m_2)} \Gamma(3/2 - \ell).$$

However we are keen to obtain all the nonleading terms and this can be accomplished by rewriting the sunset integral as

$$I_2(p) = \frac{\Gamma(2-\ell)(m_1 m_2 \Delta^2)^{\ell-3/2}}{(4\pi)^\ell \sqrt{p^2}} \int_{u_-}^{u_+} du [1 + u^2 - u\eta]^{\ell-2},$$

where the new integration variable is $u = \rho \sqrt{p^2/m_1 m_2}$, the limits are $u_+ = \frac{m_2}{(m_1+m_2)\Delta} \sqrt{\frac{p^2}{m_1 m_2}}$ and $u_- = -\frac{m_1}{(m_1+m_2)\Delta} \sqrt{\frac{p^2}{m_1 m_2}}$, and the new expansion variable is $\eta \equiv \frac{\Delta(m_1^2 - m_2^2)}{\sqrt{p^2 m_1 m_2}}$. The series in η reads

$$I_2(p) = \frac{(m_1 m_2 \Delta^2)^{\ell-3/2}}{(4\pi)^\ell \sqrt{p^2}} \sum_{n=0}^{\infty} \frac{\Gamma(2+n-\ell)}{n!} \eta^n U_n; \quad U_n \equiv \int_{u_-}^{u_+} du (1 + u^2)^{\ell-2-n} u^n.$$

In evaluating the coefficients U_n , we make use of the fact that the limits are large and we should distinguish between the cases of n even and n odd. For n even, we split the integral into

$$\begin{aligned} U_n &= \left(2 \int_0^\infty - \int_{|u_-|}^\infty - \int_{|u_+|}^\infty \right) du u^n (1 + u^2)^{\ell-2-n} \\ &= \frac{\Gamma((n+1)/2) \Gamma((3+n)/2 - \ell)}{\Gamma(2+n-\ell)} - \left(\int_{|u_-|}^\infty + \int_{|u_+|}^\infty \right) du u^{2\ell-4-n} (1 + 1/u^2)^{\ell-2-n}, \end{aligned}$$

where we may expand the latter two integrals in powers of $1/u^2$ — which provides a further sub-expansion in powers of Δ . Thus we get

$$\Gamma(2+n-\ell)U_n = \Gamma((n+1/2)\Gamma((3+n)/2-\ell) + \sum_{k=0}^{\infty} \frac{(-)^k \Gamma(2+n+k-\ell)}{k!(2\ell-3-n-2k)} \times \\ \left[|u_+|^{2\ell-3-n-2k} + |u_-|^{2\ell-3-n-2k} \right].$$

On the other hand, the case of odd n can be treated directly by expanding the integrand below in powers of $1/v$:

$$U_n = \frac{1}{2} \int_{u_-^2}^{u_+^2} dv v^{\ell-(5+n)/2} (1+1/v)^{\ell-2-n} \quad \text{so} \\ \Gamma(2+n-\ell)U_n = \sum_{k=0}^{\infty} \frac{(-)^k \Gamma(2+n+k-\ell)}{k!(2\ell-3-n-2k)} \left[|u_+|^{2\ell-3-n-2k} - |u_-|^{2\ell-3-n-2k} \right].$$

Again one encounters a sub-series in Δ . (The two series make up the ‘h-h’ and ‘p-p’ contributions of ref [3] at one stroke.) In short we see that the sunset integral, and of necessity its imaginary part, can be systematically expanded as a series in Δ which is proportional to the Q -value of the reaction by appropriately handling the Feynman parametric representation. It is possible to treat the sunset diagram with N intermediate particles in a similar way, but the method becomes rather unwieldy, which is why we turned to coordinate space methods in sections IV, V.

References

- [1] R. Hagedorn, *Nuovo Cim.* **25**, 1017 (1962); B. Almgren, *Arkiv für Fysik* **38**, 161 (1968); E. Byckling and K. Kajantie, ‘Particle Kinematics’ (Wiley, New York, 1973).
- [2] S. Groote, J.G. Körner and A.A. Pivovarov, *Nucl. Phys.* **B542**, 515 (1999).
- [3] A.I. Davydychev and V.A. Smirnov, *Nucl. Phys.* **B554**, 391 (1999).
- [4] F.A. Berends, A.I. Davydychev and N.I. Ussyukina, *Phys. Lett.* **B426**, 95 (1998).
- [5] T.W.B. Kibble, *Phys. Rev.* **117**, 1159 (1960).
- [6] S. Bauberger, F.A. Berends, M. Böhm and M. Buza, *Nucl. Phys.* **B434**, 383 (1995).
- [7] A.K. Rajantie, *Nucl. Phys.* **B480**, 729 (1996) has obtained the full $I_3(p)$ for $D = 3$.
- [8] D.A. Akyeampong and R. Delbourgo, *Nuovo Cim.* **19A**, 141 (1974); E. Mendels, *Nuovo Cim.* **45A**, 87 (1978).
- [9] F.A. Berends, M. Buza, M. Böhm and R. Scharf, *Z. Phys.* **C63**, 227 (1994).
- [10] I.S. Gradshteyn and I.M. Ryzhik, ‘Tables of Integrals, Series and Products’ (Academic Press, 1994); W. Magnus, F. Oberhettinger and R.P. Soni, ‘Formulas and Theorems for the Special Functions of Mathematical Physics’ (Springer-Verlag, New York, 1966).
- [11] E.E. Boos and A.I. Davydychev, *Theor. Math. Phys.* **89**, 1052 (1991).
- [12] M. Beneke and V.A. Smirnov, *Nucl. Phys.* **B522**, 321 (1998); see eqn (12).

- [13] S. Groote, J.G. Körner and A.A. Pivovarov, Eur. Phys. J. **C11**, 279 (1999).
- [14] S. Groote and A.A. Pivovarov, Nucl. Phys. **B50**, 459 (2000). We were completely unaware of this reference when we carried out our own research and it was only pointed out to us afterwards.
- [15] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, ‘Integrals & Series, Vol. 2 (Special Functions)’ (Gordon and Breach, 1986); see also A. Gervois and H. Navelet, J. Math. Phys. **27**, 682, 688 (1986) for additional information about integrals of products of three Bessel functions.
- [16] A.I. Davydychev and R. Delbourgo, J. Math. Phys. **39**, 4299 (1998).